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The ratios of the light quark masses

H. Leutwyler
Institut für theoretische Physik der Universität Bern
Sidlerstr. 5, CH-3012 Berne, Switzerland and

CERN, CH-1211 Geneva 23
Switzerland

Abstract

The paper collects the various pieces of information concerning the relative size of m_u , m_d and m_s . A coherent picture results, which constrains the mass ratios to a rather narrow range: $m_u/m_d = 0.553 \pm 0.043$, $m_s/m_d = 18.9 \pm 0.8$.

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1. I wish to show that recent results of chiral perturbation theory allow a rather accurate determination of the relative size of m_u , m_d and m_s . The paper amounts to an update of earlier work [1]–[6] based on the same method. Sum rules and numerical simulations of QCD on a lattice represent alternative approaches with a broader scope – they permit a determination not only of the ratios $m_u : m_d : m_s$, but also of the individual quark masses, including the heavy ones. The sum rule results for the ratios are subject to comparatively large errors [7, 8]. Concerning the lattice technique, considerable progress has been made [9, 10]. It is difficult, however, to properly account for the vacuum fluctuations generated by quarks with small masses. Further progress with light dynamical fermions is required before the numbers obtained for m_u/m_d or m_s/m_d can be taken at face value.

2. The quark masses depend on the renormalization scheme. Chiral perturbation theory treats the mass term of the light quarks, $m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s$, as a perturbation [11, 12]. It exploits the fact that, *for mass independent renormalization schemes*, the operators $\bar{u}u$, $\bar{d}d$ and $\bar{s}s$ transform as members of the representation $(3, 3^*) + (3^*, 3)$. Since all other operators with this transformation property are of higher dimension, the normalization conventions for m_u , m_d and m_s then differ only by a flavour-independent factor. The factor, in particular, also depends on the renormalization scale, but in the ratios m_u/m_d and m_s/m_d , it drops out – these represent convention-independent pure numbers.

3. The leading order mass formulae for the Goldstone bosons follow from the relation of Gell-Mann, Oakes and Renner. Disregarding the electromagnetic interaction, they read $M_{\pi^+}^2 = (m_u + m_d)B$, $M_{K^+}^2 = (m_u + m_s)B$, $M_{K^0}^2 = (m_d + m_s)B$, where the constant of proportionality is determined by the quark condensate: $B = |\langle 0 | \bar{u}u | 0 \rangle| / F_\pi^2$. Solving for the quark masses and forming ratios, this constant drops out, so that m_u/m_d and m_s/m_d may be expressed in terms of ratios of meson masses. Current algebra also shows that the mass difference between the π^+ and the π^0 is almost exclusively due to the electromagnetic interaction – the contribution generated by $m_d \neq m_u$ is of order $(m_d - m_u)^2$ and therefore tiny. Moreover, the Dashen theorem states that, in the chiral limit, the electromagnetic contributions to $M_{K^+}^2$ and to $M_{\pi^+}^2$ are the same, while the self energies of K^0 and π^0 vanish. Using these relations to correct for the electromagnetic self energies, the above

lowest order mass formulae yield [1]

$$\begin{aligned}\frac{m_u}{m_d} &= \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 0.55 \ , \\ \frac{m_s}{m_d} &= \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 20.1 \ .\end{aligned}\tag{1}$$

4. These relations and the analysis given below are based on the hypothesis that the quark condensate is the leading order parameter of the spontaneously broken symmetry. This hypothesis is questioned in refs. [13], where a more general scenario is described, referred to as generalized chiral perturbation theory: Stern and co-workers investigate the possibility that the correction of $O(m^2)$ in the expansion $M_\pi^2 = (m_u + m_d)B + O(m^2)$ is comparable with or even larger than the term that originates in the quark condensate. Indeed, the available evidence does not exclude this possibility, but a beautiful experimental proposal has been made [14]: $\pi^+\pi^-$ atoms decay into a pair of neutral pions, through the strong transition $\pi^+\pi^- \rightarrow \pi^0\pi^0$. Because the momentum transfer nearly vanishes, the decay rate is determined by the combination $a_0 - a_2$ of S-wave $\pi\pi$ scattering lengths. Since chiral symmetry implies that Goldstone bosons of zero energy do not interact, a_0, a_2 vanish in the limit $m_u, m_d \rightarrow 0$. The transition amplitude, therefore, directly measures the symmetry breaking generated by m_u, m_d . Standard chiral perturbation theory yields very sharp predictions for a_0, a_2 [15], while the generalized scenario does not [16]. A measurement of the lifetime of a $\pi^+\pi^-$ atom would thus allow us to decide whether or not the quark condensate represents the leading order parameter.

5. The contributions of first non-leading order were worked out in ref. [12]. As it turns out, the correction in the mass ratio $(M_{K^0}^2 - M_{K^+}^2)/(M_K^2 - M_\pi^2)$ is the same as the one in M_K^2/M_π^2 :

$$\begin{aligned}\frac{M_K^2}{M_\pi^2} &= \frac{\hat{m} + m_s}{m_u + m_d} \{1 + \Delta_M + O(m^2)\} \ , \\ \frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} &= \frac{m_d - m_u}{m_s - \hat{m}} \{1 + \Delta_M + O(m^2)\} \ .\end{aligned}\tag{2}$$

The quantity Δ_M accounts for the breaking of SU(3) and is related to the

effective coupling constants L_5 and L_8 :

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2} (2L_8 - L_5) + \chi \text{logs} \quad . \quad (3)$$

The term χlogs stands for the logarithms characteristic of chiral perturbation theory (for an explicit expression, see [12]). The above relations imply that the double ratio

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \quad (4)$$

is given by a ratio of meson masses, up to corrections of second order,

$$Q^2 \equiv \frac{M_K^2}{M_\pi^2} \cdot \frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2} \{1 + O(m^2)\} \quad . \quad (5)$$

The result may be visualized by plotting m_s/m_d versus m_u/m_d [17]. The constraint then takes the form of an ellipse,

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2} \left(\frac{m_s}{m_d}\right)^2 = 1 \quad , \quad (6)$$

with Q as major semi-axis, the minor one being equal to 1 (for simplicity, I have discarded the term \hat{m}^2/m_s^2 , which is numerically very small).

6. The meson masses occurring in the double ratio (5) refer to pure QCD. Correcting for the electromagnetic self energies with the Dashen theorem, the quantity Q becomes

$$Q_D^2 = \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4 M_{\pi^0}^2 (M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)} \quad . \quad (7)$$

Numerically, this yields $Q_D = 24.2$. For this value of the semi-axis, the ellipse passes through the point specified by Weinberg's mass ratios, which correspond to $\Delta_M=0$, $Q=Q_D$.

The Dashen theorem is subject to corrections from higher order terms in the chiral expansion, which are analysed in several recent papers. Donoghue, Holstein and Wyler [18] estimate the contributions arising from vector meson exchange and conclude that these give rise to large corrections, increasing the value $(M_{K^+} - M_{K^0})_{e.m.} = 1.3$ MeV predicted by Dashen to 2.3 MeV. According to Baur and Urech [19], however, the model used is in conflict with chiral

symmetry: although the perturbations due to vector meson exchange are enhanced by a relatively small energy denominator, chiral symmetry prevents them from being large. In view of this, it is puzzling that Bijnens [20], who evaluates the self energies within the model of Bardeen et al., finds an even larger effect, $(M_{K^+} - M_{K^0})_{e.m.} \simeq 2.6 \text{ MeV}$. The implications of the above estimates for the value of Q are illustrated on the r.h.s. of fig. 1.

Recently, the electromagnetic self energies have been analysed within lattice QCD [10]. The result of this calculation, $(M_{K^+} - M_{K^0})_{e.m.} = 1.9 \text{ MeV}$, indicates that the corrections to the Dashen theorem are indeed substantial, although not quite as large as found in refs. [18, 20]. The uncertainties of the lattice result are of the same type as those occurring in direct determinations of the quark masses with this method. The mass difference between K^+ and K^0 , however, is predominantly due to $m_d > m_u$, not to the e.m. interaction. An error of 20% in the self energy affects the value of Q by only about 3%. The terms neglected when evaluating Q^2 with the meson masses are of order $(M_K^2 - M_\pi^2)^2/M_0^4$, where M_0 is the mass scale relevant for the exchange of scalar or pseudoscalar states, $M_0 \simeq M_{a_0} \simeq M_{\eta'}$ [4]. The corresponding error in the result for Q is also of the order of 3% – the uncertainties in the value $Q = 22.8$ that follows from the lattice result are significantly smaller than those obtained for the quark masses with the same method.

7. The isospin-violating decay $\eta \rightarrow 3\pi$ allows one to measure the semi-axis in an entirely independent manner [21]. The transition amplitude is much less sensitive to the uncertainties associated with the electromagnetic interaction than the K^0 – K^+ mass difference: the e.m. contribution is suppressed by chiral symmetry and is negligibly small [22]. The decay $\eta \rightarrow 3\pi$ thus represents a sensitive probe of the symmetry breaking generated by $m_d - m_u$. It is convenient to write the decay rate in the form $\Gamma_{\eta \rightarrow \pi^+ \pi^- \pi^0} = \Gamma_0 (Q_D/Q)^4$, where Q_D is specified in eq. (7). As shown in ref. [21], chiral perturbation theory to one loop yields a parameter-free prediction for the constant Γ_0 . Updating the value of F_π , the numerical result reads $\Gamma_0 = 168 \pm 50 \text{ eV}$. Although the calculation includes all corrections of first non-leading order, the error bar is large. The problem originates in the final state interaction, which strongly amplifies the transition probability in part of the Dalitz plot. The one-loop calculation does account for this phenomenon, but only to leading order in the low energy expansion. The final state interaction is analysed more accurately in two recent papers [23, 24], which exploit the fact that analyticity and unitarity determine the amplitude up to a few subtraction constants.

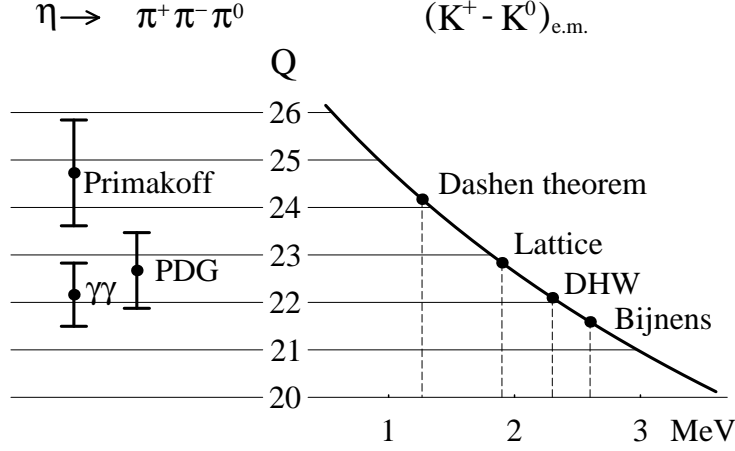


Figure 1: The l.h.s. indicates the values of Q corresponding to the various experimental results for the rate of the decay $\eta \rightarrow \pi^+ \pi^- \pi^0$. The r.h.s. shows the results for Q obtained with four different theoretical estimates for the electromagnetic self energies of the kaons.

For these, the corrections to the current algebra predictions are small, because they are barely affected by the final state interaction. Although the dispersive framework used in the two papers differs, the results are nearly the same: while Kambor, Wiesendanger and Wyler obtain $\Gamma_0 = 209 \pm 20$ eV, we get $\Gamma_0 = 219 \pm 22$ eV. This shows that the theoretical uncertainties of the dispersive calculation are small.

Unfortunately, the experimental situation is not clear [25]. The value of $\Gamma_{\eta \rightarrow \pi^+ \pi^- \pi^0}$ relies on the rate of the decay into two photons. The two different methods of measuring $\Gamma_{\eta \rightarrow \gamma\gamma}$ (photon-photon collisions and Primakoff effect) yield conflicting results. While the data based on the Primakoff effect are in perfect agreement with the number $Q = 24.2$, which follows from the Dashen theorem, the $\gamma\gamma$ data yield a significantly lower result (see l.h.s. of fig. 2). The statistics is dominated by the $\gamma\gamma$ data. Using the overall fit of the Particle Data Group, $\Gamma_{\eta \rightarrow \pi^+ \pi^- \pi^0} = 283 \pm 28$ eV [25], and adding errors quadratically, we obtain $Q = 22.7 \pm 0.8$, to be compared with the result $Q = 22.4 \pm 0.9$ given in ref. [23]. With this value of Q , the low energy theorem (5) implies that the electromagnetic self energy amounts to $(M_{K^+} - M_{K^0})_{e.m.} = 2$ MeV, to within an uncertainty of the order of 20%, in agreement with the lattice result. I conclude that, within the remarkably

small errors of the individual determinations, the two different methods of measuring Q are consistent with each other, but repeat that one of these relies on the lifetime of the η , where the experimental situation is not satisfactory.

8. Kaplan and Manohar [17] pointed out that a change in the quark masses of the form $m'_u = m_u + \alpha m_d m_s$ (cycl. $u \rightarrow d \rightarrow s \rightarrow u$) may be absorbed in a change of the effective coupling constants L_6, L_7, L_8 . The results obtained with the effective Lagrangian for the meson masses, scattering amplitudes and matrix elements of the vector and axial currents are invariant under the operation. Conversely, since the ratios m_u/m_d and m_s/m_d do not remain invariant, they cannot be determined with the experimental low energy information concerning these observables¹. In particular, phenomenology by itself does not exclude the value $m_u=0$, widely discussed in the literature [26], as a possible solution of the strong CP puzzle.

We are not dealing with a symmetry of QCD, nor is the effective Lagrangian intrinsically ambiguous: even at the level of the effective theory, the predictions for the matrix elements of the scalar and pseudoscalar operators are not invariant under the above transformation. Since an experimental probe sensitive to these is not available, however, the size of the correction Δ_M in eq. (2) cannot be determined on purely phenomenological grounds – theoretical input is needed for this purpose. In the following, I use the $1/N_c$ expansion and the requirement that SU(3) represents a decent approximate symmetry. For a more detailed discussion of the issue, I refer to [6].

9. The problem disappears in the large- N_c limit, because the transformation $m'_u = m_u + \alpha m_d m_s$ violates the Zweig rule [3, 4]. For $N_c \rightarrow \infty$, the quark loop graph that gives rise to the U(1) anomaly is suppressed, so that QCD acquires an additional symmetry, whose spontaneous breakdown gives rise to a ninth Goldstone boson, the η' . The implications for the effective Lagrangian are extensively discussed in the literature, and the leading terms in the expansion in powers of $1/N_c$ were worked out long ago [27]. More recently, the analysis was extended to first non-leading order, accounting for all terms which are suppressed either by one power of $1/N_c$ or by one power of the quark mass matrix [28]. This framework leads to the bound

$$\Delta_M > 0 \quad , \quad (8)$$

¹The transformation maps the elliptic constraint onto itself: to first order in isospin breaking, the quantity $1/Q^2$ may equivalently be written as $(m_d^2 - m_u^2)/(m_s^2 - m_d^2)$, and the differences $m_d^2 - m_u^2, m_s^2 - m_d^2, m_u^2 - m_s^2$ are invariant.

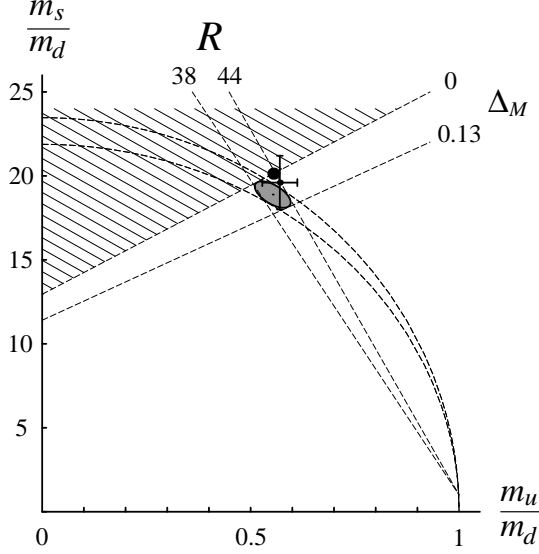


Figure 2: Quark mass ratios. The dot corresponds to Weinberg's values, while the cross represents the estimates given in ref. [2]. The hatched region is excluded by the bound $\Delta_M > 0$. The error ellipse shown is characterized by the constraints $Q = 22.7 \pm 0.8$, $\Delta_M > 0$, $R < 44$, which are indicated by dashed lines.

which excludes the hatched region in fig. 2. Since the Weinberg ratios correspond to $\Delta_M = 0$, they are located at the boundary of this region. In view of the elliptic constraint, the bound in particular implies $m_u/m_d \gtrsim \frac{1}{2}$ and thus excludes a massless u -quark.

10. An upper limit for m_u/m_d may be obtained from the branching ratio $\Gamma_{\psi' \rightarrow \psi \pi^0} / \Gamma_{\psi' \rightarrow \psi \eta}$. Disregarding electromagnetic contributions [29], the ratio of transition amplitudes is proportional to $(m_d - m_u)/(m_s - \hat{m})$:

$$\frac{T_{\psi' \rightarrow \psi \pi^0}}{T_{\psi' \rightarrow \psi \eta}} = \frac{3\sqrt{3}}{4R} (1 + \Delta_{\psi'}) \quad , \quad \frac{1}{R} \equiv \frac{m_d - m_u}{m_s - \hat{m}} \quad .$$

SU(3) predicts that, for quarks of equal mass, $\Delta_{\psi'}$ vanishes: this term represents an SU(3)-breaking effect of order $m_s - \hat{m}$. The data on the branching ratio imply $R = (31 \pm 4) (1 + \Delta_{\psi'})$, where the given error only accounts for the experimental accuracy. The breaking of SU(3) is analysed in ref. [29], on the basis of the multipole expansion. The calculation yields a remarkably small

result for $\Delta_{\psi'}$, indicating a value of R close to 31, but the validity of the multipole expansion for the relevant transition matrix elements is doubtful [30]. Moreover, fig. 2 shows that the result of this calculation is in conflict with the large- N_c bound. Since the quark mass ratios given in refs. [5] rely on the value of R obtained in this way, they face the same objections.

At the present level of theoretical understanding, the magnitude of $\Delta_{\psi'}$ is too uncertain to allow a determination of R , but I do not see any reason to doubt that SU(3) represents a decent approximate symmetry also for charmonium. The scale of first order SU(3) breaking effects such as Δ_M , $(F_K - F_\pi)/F_\pi$ or $\Delta_{\psi'}$ is set by $(M_K^2 - M_\pi^2)/M_0^2 \simeq 0.25$. Indeed, a correction of this size would remove the discrepancy with the large- N_c bound. Large values of R , on the other hand, are inconsistent with the eightfold way. As a conservative upper limit for the breaking of SU(3), I use $|\Delta_{\psi'}| < 0.4$. Expressed in terms of R , this implies $R < 44$. The value $m_s/\hat{m} = 29 \pm 7$, obtained by Bijmens, Prades and de Rafael with QCD sum rules [8], yields an independent check: the lower end of this interval corresponds to $\Delta_M < 0.17$. Figure 2 shows that this constraint also restricts the allowed region to the right and is only slightly weaker than the above condition on R .

11. The net result for the quark mass ratios is indicated by the shaded error ellipse in fig. 2, which is defined by the following three constraints: (i) On the upper and lower sides, the ellipse is bounded by the two dashed lines that correspond to $Q = 22.7 \pm 0.8$. (ii) To the left, it touches the hatched region, excluded by the large- N_c bound. (iii) On the right, I use the upper limit $R < 44$, which follows from the observed value of the branching ratio $\Gamma_{\psi' \rightarrow \psi \pi^0} / \Gamma_{\psi' \rightarrow \psi \eta}$. The corresponding range of the various parameters of interest is

$$\begin{aligned} \frac{m_u}{m_d} &= 0.553 \pm 0.043 \quad , \quad \frac{m_s}{m_d} = 18.9 \pm 0.8 \quad , \quad \frac{m_s}{m_u} = 34.4 \pm 3.7 \quad , \\ \frac{m_s - \hat{m}}{m_d - m_u} &= 40.8 \pm 3.2 \quad , \quad \frac{m_s}{\hat{m}} = 24.4 \pm 1.5 \quad , \quad \Delta_M = 0.065 \pm 0.065 \quad . \end{aligned}$$

While the central value for m_u/m_d happens to coincide with the leading order formula, the one for m_s/m_d turns out to be slightly smaller. The difference, which amounts to 6%, originates in the fact that the available data on the η lifetime as well as the lattice result for the electromagnetic self energies of the kaons imply a somewhat smaller value of Q than what is predicted by the Dashen theorem, in agreement with ref. [5].

The result for the ratio of isospin- to SU(3)-breaking mass differences, $R = 40.8 \pm 3.2$, confirms the early determinations described in [2]. As shown there, the mass splittings in the baryon octet yield three independent estimates of R , i.e. 51 ± 10 ($N-P$), 43 ± 4 ($\Sigma^- - \Sigma^+$) and 42 ± 6 ($\Xi^- - \Xi^0$)². These numbers are perfectly consistent with the value given above. A recent reanalysis of ρ - ω mixing [31] leads to $R = 41 \pm 4$ and thus corroborates the picture further.

I find it remarkable that, despite the problems generated by the determinant of the Dirac operator for quark masses of realistic size, the lattice results for the mass ratios are quite close to the above numbers. The most recent values are $m_u/m_d = 0.512 \pm 0.006$, $(m_d - m_u)/m_s = 0.0249 \pm 0.0003$, where the error only accounts for the statistical noise [10]. They correspond to $Q = 22.9$, $\Delta_M = 0$, $R = 38.6$ – the place where the error ellipse shown in fig. 2 touches the large- N_c bound.

Finally, I use the value of m_s obtained with QCD sum rules [7, 8] as an input and calculate m_u and m_d with the above ratios. The result for the running masses in the $\overline{\text{MS}}$ scheme at scale $\mu = 1 \text{ GeV}$ reads

$$m_u = 5.1 \pm 0.9 \text{ MeV} \quad , \quad m_d = 9.3 \pm 1.4 \text{ MeV} \quad , \quad m_s = 175 \pm 25 \text{ MeV} \quad .$$

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²Note that, in this case, the expansion contains terms of order $m^{\frac{3}{2}}$, which do play a significant role numerically. The error bars represent simple rule-of-thumb estimates, indicated by the noise visible in the calculation. For details see ref. [2].

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